Analysis and Synthesis of Tapered Microstrip Transmission Lines

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Abstract—**The input voltage reflection coefficient on a tapered microstrip transmission line is analyzed and a new Fourier transform pair is derived which introduces echo time and frequency as variables. Calculations of the input reflection coefficient take into account the frequency dispersion-characteristics of the effective permittivity. An efficient method is proposed for analysis and synthesis of microstrip tapers and numerical results are presented for microstrip exponential tapers and microstrip Tchebycheff tapers. The present paper is the first one shown the frequency-dependent characteristics of reflection coefficients.**

I. Introduction

Tapered microstrip transmission lines are an increasingly important part in the design of matching networks, filters, couplers, circulators, etc., in MICs.

Consider the tapered microstrip transmission line, supporting a non-TEM mode, shown in Fig. 1 which use as a transformer to match a line of impedance Z_1 to a load of impedance Z_2. When an electrical signal of frequency ω is transmitted in the $+z$ direction, a portion of the signal is reflected back to the sending point $z = 0$. In analyzing the reflection coefficient, the phase constant β and characteristic impedance Z at position z are approximated as those of a uniform line having the same cross-sectional dimensions as the taper has at that position. The phase constant β along the taper is expressed in terms of the effective permittivity $\epsilon_{\text{eff}}(z, \omega)$ as shown by (1) in the next section. The effective permittivity varies along the taper because of a variation of the width of strip conductor along the taper and has the frequency dispersion-characteristics.

In previous analysis, the characteristic impedance $Z(z)$ depends on only position z. In this case, it is well-known (see [1]–[7] and references therein) that the Riccati equation holds for the voltage reflection coefficients ρ of the taper. For the case of $\rho^2 << 1$ and ϵ_{eff} = constant along the taper, the reflection coefficient and the Fourier transform pair were given in [6]. For the case of $\rho^2 << 1$ and $\epsilon_{\text{eff}} = \epsilon_{\text{eff}}(z)$, the reflection coefficients were calculated numerically [1] and the pair of variables p and u used in [6] were modified in [2].

The present paper derives another Fourier transform pair introducing an echo time and frequency as variables

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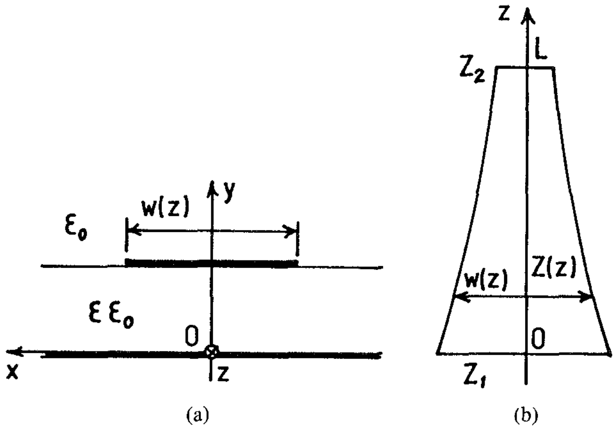

Fig. 1. Tapered microstrip transmission line. (a) Cross-section. (b) z-dependent configuration of strip conductor.

for the case of $\epsilon_{\text{eff}} = \epsilon_{\text{eff}}(z)$. Furthermore, the reflection coefficients are calculated numerically for the case of $\epsilon_{\text{eff}} = \epsilon_{\text{eff}}(z, \omega)$. An efficient method for analysis and synthesis is derived and is applied to microstrip exponential tapers and microstrip Tchebycheff tapers to show the validity of the present method for the case of position and frequency-dependent effective permittivities.

II. Fourier Transform Pair Having Echo Time and Frequency as Variables

The phase constant β of the tapered line shown in Fig. 1 is expressed in terms of a phase velocity v as follows:

$$\beta = \omega / v(z, \omega) \tag{1}$$

where

$$v(z, \omega) = c / \sqrt{\epsilon_{\text{eff}}(z, \omega)} \tag{2}$$

and c denotes velocity of light in free space.

Let a new variable τ be introduced as follows:

$$\tau = 2 \int_0^z \frac{1}{v(z, \omega)} dz, \tag{3}$$

where τ equals the travelling time for the case in which an electrical signal of angular frequency ω transmitted at $z = 0$ travels along the line, reflects at position z, and returns to the sending point at $z = 0$. Therefore, let τ be called the echo time.

Using the variable τ, the Riccati equation gives the input reflection coefficient at $z = 0$, after some manual calculation:

$$\rho = \int_0^{\tau_L} e^{-j\omega\tau} \frac{1}{2} \frac{d \ln (Z/Z_1)}{d\tau} d\tau \tag{4}$$

where

$$\tau_L = \int_0^L \frac{2}{v(z, \omega)} dz. \tag{5}$$

This equation may be interpreted as the Fourier transform of a function $(1/2)\, d \ln (Z/Z_1)/d\tau$, which is zero outside the range $0 \leqq \tau \leqq \tau_L$. It is to be noted that no restrictions of any kind have been imposed except $\rho^2 << 1$.

For only the case of $v(z)$ with $\epsilon_{\text{eff}}(z)$, no dispersion, two variables are independent each other and so we can consider the inverse Fourier integral as follows:

$$\frac{1}{2} \frac{d \ln (Z/Z_1)}{d\tau} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega\tau} \rho \, d\omega. \tag{6}$$

Equations (4) and (6) form a simple Fourier transform pair.

III. Reflection Coefficient of Exponential Taper

Let us consider the exponential tapered microstrip transmission line of length L as an example of the taper shown in Fig. 1. For a match between two impedances, Z_1 and Z_2, the characteristic impedance along the line can be expressed as follows:

$$Z(z) = Z_1 \exp \{(z/L) \ln (Z_2/Z_1)\}. \tag{7}$$

Substituting (7) into (4) gives the following equation:

$$\rho = \int_0^{\tau_L} e^{-j\omega\tau} \frac{c \ln (Z_2/Z_1)}{2L \sqrt{\epsilon_{\text{eff}}(z, \omega)}} \exp \{(z/L) \ln (Z_2/Z_1)\} \, d\tau. \tag{8}$$

If the configuration of the taper, that is the shape ratio w/h at position z along the taper, is given, the value of $\epsilon_{\text{eff}}(z, \omega)$ can be obtained using the dispersion formula proposed in [10].

The relation between w/h and z/L, in turn, can be determined from characteristic impedance Z in (7) and the following relation for the characteristic impedance Z is given by

$$Z = \eta / \{(C_o(w/h)/\epsilon_o) \sqrt{\epsilon_{\text{eff}}(w/h, 0)}\} \tag{9}$$

where $\eta\ (= \sqrt{\mu_o/\epsilon_o})$ denotes the intrinsic impedance of free space and $C_o(w/h)$ denotes the line capacitance per unit length for the case of $\epsilon = 1$ and shape ratio w/h. Using the results tabulated in [8] an approximate formula of $C_o(w/h)/\epsilon_o$ for the case of $0.1 \leqq w/h \leqq 0.7$ can be given by

$$C_o(w/h)/\epsilon_o = 0.4109 + \sqrt{5.940w/h + 0.4631}. \tag{10a}$$

This small range for w/h was taken only for a simplicity in the calculation since the dispersion formula for $\epsilon_{\text{eff}}(z, \omega)$ proposed in [10] was given in different forms for the cases of w/h narrower and wider than 0.7. Also an approximate formula for $\epsilon_{\text{eff}}(w/h, 0)$ was given in [8]. For $\epsilon = 8$ and $w/h \leqq 4.4$, it is given as

$$\begin{aligned} \epsilon_{\text{eff}}(w/h, 0) = 4.5 + 1.832 \exp [&0.9282 \log_{10} A \\ &- 0.3367 \{\log_{10} A\}^2 - 0.3189 \{\log_{10} A\}^3 \\ &- 0.0615\{\log_{10} A\}^4] \end{aligned} \tag{10b}$$

where

$$A = \log_{10} (w/h/4.4).$$

For the case of $\epsilon = 8$, $Z_1 = 63.58\ \Omega$ ($w/h = 0.7$ at $z = 0$), $Z_2 = 117.99\ \Omega$ ($w/h = 0.1$ at $z = L$), the resulting relation between w/h and z/L is shown by the dashed line in Fig. 7.

At this stage, all quantities needed to calculate numerically the reflection coefficient $\rho_i (= |\rho|)$ based on (4) are known. Results obtained using these quantities are shown by solid lines for parameter $L/h = 1, 2, 5, 10$ in Fig. 2. These curves denote the reflection coefficients for the case where the effective permittivity depends on both z and ω. These results are the first ones showing the frequency-dependent characteristics of the reflection coefficients.

The dot-dashed line shown in Fig. 2 denote the reflection coefficients obtained by approximating the effective permittivities by $\epsilon_{\text{eff}}(w/h, 0)$, no dispersion. The dot-dashed lines for all different $L/h (= 1, 2, 5, 10)$ are nearly coincident and cannot be distinguished from each other in Fig. 2. The method of Pramanick and Bhartia [1] for the case of $\epsilon_{\text{eff}}(w/h, 0)$ gives results in agreement with the dot-dashed line of Fig. 2.

Next, let us consider the exponential tapered microstrip line without substrate, that is $\epsilon = 1$, where the impedance Z along the taper is identical to the one shown in (7). For this case, (4) can be evaluated analytically to give the following reflection coefficient:

$$\rho_i = (1/2) \ln (Z_2/Z_1)\, |\sin \theta_L / \theta_L| \tag{11}$$

where θ_L denotes the electrical length and is given as

$$\theta_L = \omega \tau_L \tag{12}$$

$$\tau_L = 2L/c. \tag{13}$$

The results for $L/h = 1, 2, 5, 10$ are shown by the dashed lines in Fig. 2. In these cases the curves for different L/h are identical.

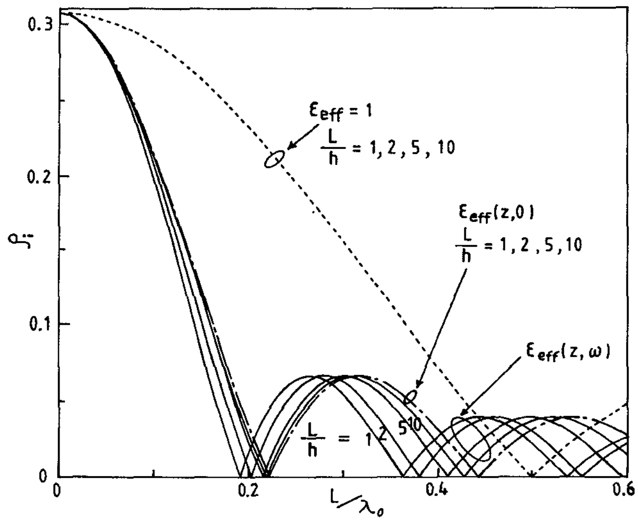

Fig. 2. Reflection coefficients for exponential microstrip tapers ($\epsilon = 8$, $Z_1 = 63.58$, $Z_2 = 117.99$). Parameter denotes the normalized taper length L/h. —— for $\epsilon_{eff}(z, \omega)$. — — — for $\epsilon_{eff}(z, 0)$. - - - - - - for $\epsilon_{eff} = 1$. The calculating method by Pramanick and Bhartia [1] gives the same curves to the ones shown by the dot-dashed line.

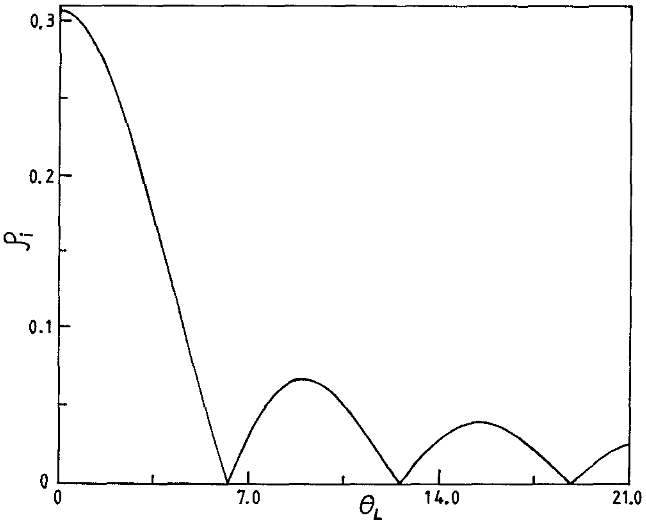

Fig. 3. Reflection coefficients obtained taking abscissa in Fig. 2 by θ_L. The curves for all cases shown in Fig. 2 are in extremely good agreement and cannot be distinguished from each other in the present figure. $\rho_i = 0$ at $\theta_L = 2n\pi$ ($n = 1, 2, 3, \cdots$).

IV. Electrical Length Matched Procedure

In general, the electrical length θ_L can be obtained from equation (12) with echo time τ_L given by (5). Fig. 3 illustrates the reflection coefficients for all curves shown in Fig. 2, where the electrical length θ_L is the abscissa. All curves are nearly coincident and cannot be distinguished from each other in Fig. 3.

The characteristics of reflection coefficients versus electrical length are the same for all cases where the variation of the normalized characteristic impedances $Z(z)/Z_1$ versus normalized position z/L are same. This is valid even if the physical configuration of taper, the physical length of taper, the operating frequency, the relative permittivity of substrate, and the frequency-dependence of effective permittivity are different between the two cases.

As a consequence, an efficient procedure can be applied to the analysis and the synthesis of a taper. The main tasks can be carried out for the cases of $\epsilon = 1$, in which both analysis and synthesis are simple because of $\epsilon_{eff} = 1$ over the taper. A flow chart for this procedure is shown in Fig. 4. The procedure is permitted only in the direction of the arrow. Let it be called an electrical length matched procedure. The columns labeled by $\epsilon_{eff} = 1$ and $\epsilon_{eff}(z, \omega)$ in Fig. 4 denote the cases of tapers with substrate of relative permittivities with $\epsilon = 1$ and $\epsilon \neq 1$, respectively. Step V1 denote the z/L-dependence of the shape ratio w/h for the taper of $\epsilon = 1$. Similarly, steps V2, V3, and V4 denote the z/L-dependence of the normalized characteristic impedance $Z(z)/Z_1$, the characteristics of the reflection coefficient ρ_i versus normalized frequency L/λ_0, and the characteristics of the reflection coefficient ρ_i versus electrical length θ_L, respectively.

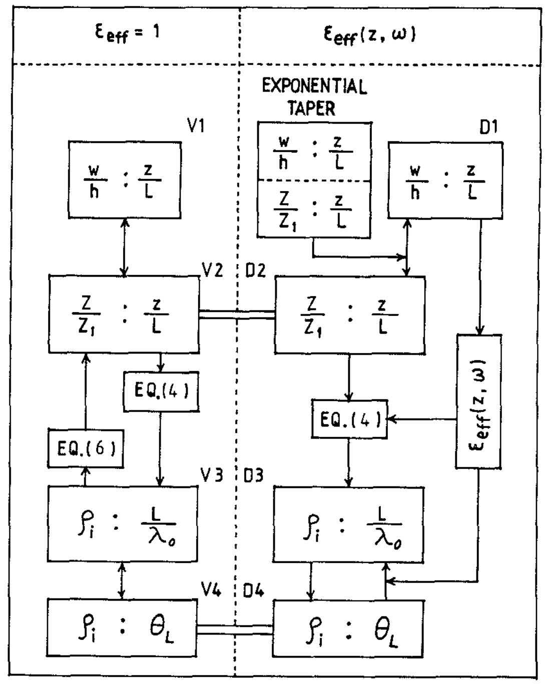

Fig. 4. Flow chart of electrical length matched procedure.

On the other hand, steps D1, D2, D3, and D4 for the cases of the taper with $\epsilon \neq 1$ correspond to steps V1, V2, V3, and V4 for the cases of the taper of $\epsilon = 1$.

Two solid lines connecting steps V2 and D2 denote letting the z/L-dependences of the normalized characteristic impedance $Z(z)/Z_1$ be matched between both steps. This matching gives the same characteristics of the reflection coefficient ρ_i versus electrical length θ_L at two steps V4 and D4. Conversely, the matching at two steps V4 and D4 gives the same z/L-dependence of the normalized characteristic impedance $Z(z)/Z_1$ at two steps V2 and D2.

For the taper of $\epsilon = 1$, we can obtain any steps VI (I

$\neq$ J) if an arbitrary step VJ is given because all steps have the arrows of both directions. For the taper of $\epsilon \neq 1$, the direct procedure from step D4 to step D2 is forbidden because there is no arrow in that direction. This is due to the non-existence of the inverse Fourier transform, for the case of effective permittivity with dispersion.

However, we can obtain step D2 by using other indirect procedure D4 = V4 ⇒ V3 ⇒ V2 = D2 when D4 is given. In the next section, let us consider a Tchebycheff taper as the example of this procedure and use it as a transformer connecting a line of impedance Z_1 to a load of impedance Z_2.

V. Analysis and Synthesis of Tchebycheff Taper

Let the Tchebycheff taper be composed of a tapered microstrip transmission line with $\epsilon = 8$, $Z_1 = 63.58\ \Omega$ ($w/h = 0.7$ at $z = 0$), $Z_2 = 117.99\ \Omega$ ($w/h = 0.1$ at $z = L$). The equality between steps V4 and D4 in the present electrical length matched procedure (Fig. 4) proposes us that the reflection coefficient for this taper, which supports a non-TEM mode can be expressed using the frequency response for a taper, which supports a TEM mode, treated previously by Collin [7], as follows:

$$\rho = \frac{1}{2} e^{-j\omega\tau_L/2} F(\omega) \tag{14}$$

where

$$F(\omega) = \ln\left(\frac{Z_2}{Z_1}\right) \frac{\cos\left(\tau_L \sqrt{\omega^2 - \omega_0^2}/2\right)}{\cosh\left(\omega_0 \tau_L/2\right)} \tag{15}$$

$$\rho_i = |\rho| \tag{16}$$

$$\rho_m = (1/2) \ln (Z_2/Z_1)/\cosh (\omega_0 \tau_L/2). \tag{17}$$

The ρ_m in (17) denotes the tolerable reflection coefficient. The ω_0 denotes the cutoff angular frequency of the taper for this ρ_m.

The characteristics of the reflection coefficient, ρ_i, versus electrical length, $\theta_L (= \omega\tau_L)$, can be obtained using equation (16) with a cutoff electrical length, $\omega_0\tau_L$, yielding a specified ρ_m in (17). This means that step D4 and namely step V4 can be given. We can obtain straightforward the characteristics of reflection coefficient, ρ_i, versus normalized frequency, L/λ_0, by dividing electrical length, θ_L, at step V4 by 4π since $\epsilon_{\text{eff}} = 1$. Fig. 5 shows the results for $\rho_m = 0.1, 0.05, 0.01$ as well as the result for the exponential taper for a comparison. Multiplying the abscissa in Fig. 5 by 4π gives the characteristics of reflection coefficient ρ_i versus electrical length θ_L.

Fig. 6 shows the z/L-dependence of characteristic impedance Z/Z_1 for microstrip Tchebycheff tapers obtained by the calculation in accordance with the indirect procedure (D4 = V4 ⇒ V3 ⇒ V2 = D2) shown in Fig. 4.

Now, the characteristic impedances have the same values at positions which have the same shape ratio w/h and the same relative permittivity. This means that the curves of the z/L-dependence of characteristic impedance and

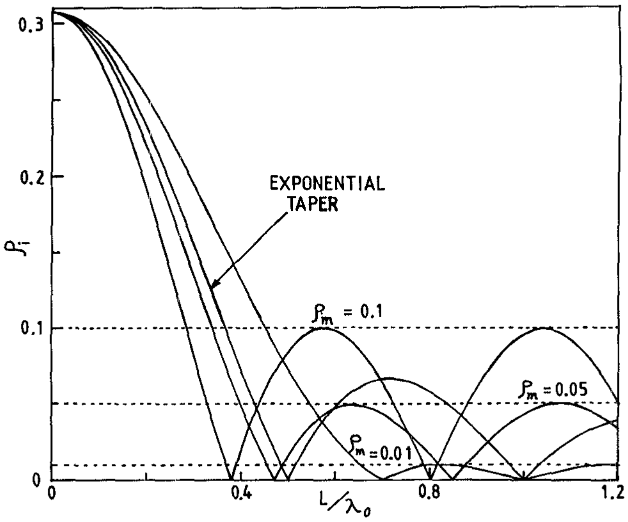

Fig. 5. Reflection coefficients for the exponential microstrip taper and the microstrip Tchebycheff tapers made of microstrip transmission lines of $\epsilon_{\text{eff}} = 1$ (that is $\epsilon = 1$).

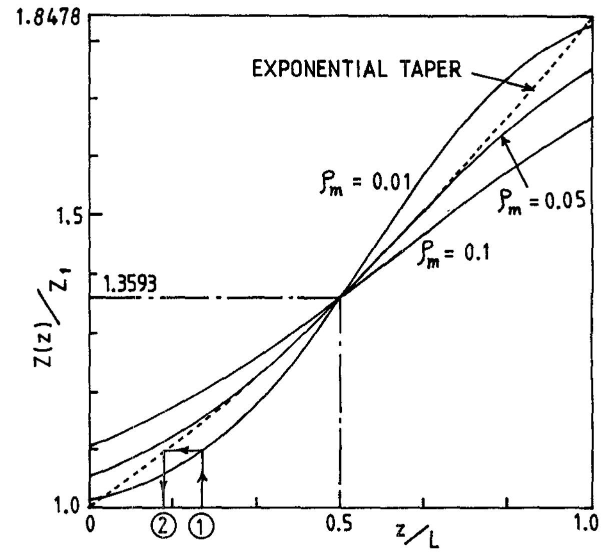

Fig. 6. The z-dependence of normalized characteristic impedances $Z(z)/Z_1$ for exponential microstrip taper and microstrip Tchebycheff tapers. The ρ_m denotes the tolerable reflection coefficient.

the configuration for the exponential microstrip taper obtained in the Section III can be used in the procedure between steps D1 and D2. The configuration (step D1) of the Tchebycheff tapers shown in Fig. 7 was obtained by the method based on this idea. Let us show its procedure by the graphic explanation for the case of $\rho_m = 0.01$ in Figs. 6 and 7. The shape ratio w/h (mark ③ in Fig. 7) for the z/L (mark ①) given in Fig. 6 can be obtained by the procedure of ① ⇒ ② in Fig. 6 and subsequently ② ⇒ ③ in Fig. 7. The arrow pulled down from the point with mark ③ in Fig. 7 gives the z/L given first. The characteristic impedance has discontinuous jumps at both ends. The characteristic impedance at the center of taper is equal to geometric mean between Z_1 and Z_2. Fig. 8 shows the reflection coefficients (step D3) for microstrip

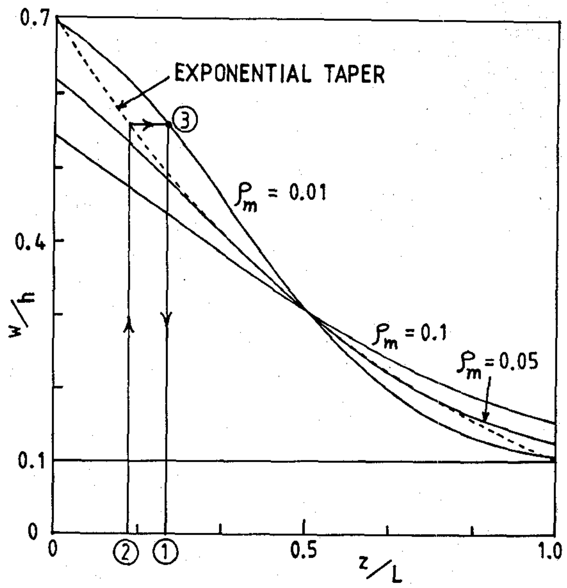

Fig. 7. The z-dependent configuration of microstrip exponential taper and Tchebycheff tapers ($\epsilon = 8$).

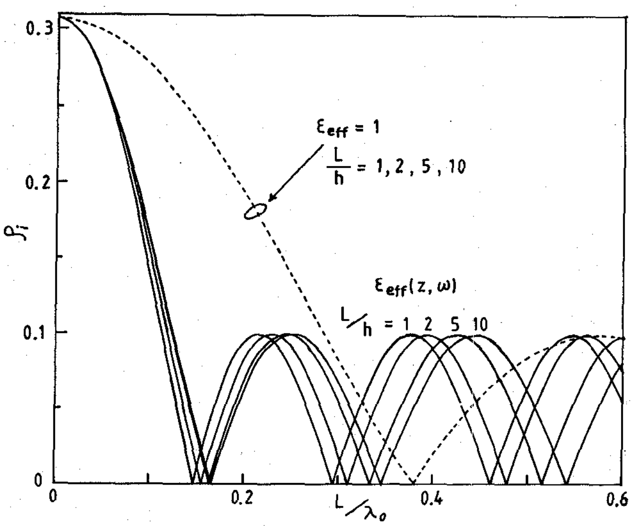

Fig. 8. Reflection coefficients for microstrip Tchebycheff tapers ($\epsilon = 8$) of configuration $\rho_m = 0.1$ shown in Fig. 7. Parameter denotes the normalized taper length L/h. —— for $\epsilon_{eff}(z, \omega)$. ---- for $\epsilon_{eff} = 1$ (this curve is for $\epsilon = 1$ and is shown for a comparison).

Tchebycheff taper obtained taking abscissa θ_L in the reflection coefficient characteristics (step D4) by L/λ_0 using $\theta_L = \omega\tau_L$ with τ_L of the integration (5) substituted with $\epsilon_{eff}(z, \omega)$ for the configuration of the taper shown in Fig. 7. These results are the first to show the dispersion-dependent characteristics of the reflection coefficients for the Tchebycheff microstrip tapers. The dashed line denotes the case of $\epsilon = 1$ ($\epsilon_{eff} = 1$) for a comparison.

VI. Conclusions

Introducing an echo time, a new Fourier transform pair has been derived for the analysis of the input reflection coefficient of a microstrip taper. Values of the reflection coefficient of microstrip exponential tapers were obtained treating the effective permittivity as being a function of position and operating frequency. This is the first time that the frequency-dependent characteristics of the effective permittivity have been presented. An efficient method for the analysis and synthesis of Tchebycheff tapers was proposed. Numerical results using this method were presented.

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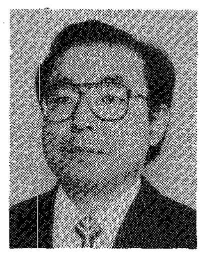

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